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An Example of a von Neumann Algebra of Global Type II

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An example of a von Neumann algebra which acts on a separable Hilbert space and is a noncentrally smooth global factor of global type II_1 has been constructed.

Two of the outstanding questions concerning von Neumann algebras are the following: Does there exist a von Neumann algebra of global type II, and is every von Neumann algebra which acts on a separable Hilbert space centrally smooth? The purpose of this note is to construct an example which will answer both of these questions, the first in the affirmative and the second in the negative. In view of E. J. Woods' recent constructive proof of the non-smoothness of the Borel space of spatial isomorphism classes of factors which act on a given separable Hilbert space [6], it is hardly surprising that one can construct an example of a non-centrally smooth von Neumann algebra. That this example is of global type II is a bit surprising, though.

Before giving the example it will be convenient to introduce some notation and recall some of the terminology of [3]. Fix once and for all a complex, separable, infinite-dimensional Hilbert space $\mathscr{H}$. Let $\mathrm{fac}(\mathscr{H})$ denote the standard Borel space consisting of all the factors on $\mathscr{H}$ with the Borel structure defined by Effros in [2], let $\mathrm{fac}(\mathscr{H})^\wedge$ denote the Borel space of spatial isomorphism classes of factors on $\mathscr{H}$ with the quotient Borel structure, and let $\pi : \mathrm{fac}(\mathscr{H}) \to \mathrm{fac}(\mathscr{H})^\wedge$ denote the quotient map. Let $\mathfrak{A}$ be a von Neumann algebra. Suppose that μ is a finite nonzero Borel measure on a standard Borel space Γ and that $\alpha \mapsto \mathfrak{A}(\alpha)$ is a Borel map from Γ into $\mathrm{fac}(\mathscr{H})$ such that $\mathfrak{A}$ is spatially isomorphic to the direct integral of the field $\alpha \mapsto \mathfrak{A}(\alpha)$ over Γ with respect to μ. It is not hard to show that whether or not the quotient Borel measure on $\mathrm{fac}(\mathscr{H})^\wedge$ determined by μ and

$\alpha \mapsto \pi(\mathfrak{A}(\alpha))$ is supported by a Borel set which is countably separated in its relative Borel structure depends only on $\mathfrak{A}$; if this is the case, then $\mathfrak{A}$ is said to be *centrally smooth*. Two central projections E and F of $\mathfrak{A}$ are said to be *globally equivalent* if $\mathfrak{A}_E$ and $\mathfrak{A}_F$ are spatially isomorphic. The lattice of central projections of $\mathfrak{A}$ together with this equivalence relation is a dimension lattice in the sense of Loomis [4], the so-called *global dimension lattice* of $\mathfrak{A}$. The properties of this dimension lattice are referred to as the global properties of $\mathfrak{A}$ (e.g., $\mathfrak{A}$ is called a global factor if the global dimension lattice of $\mathfrak{A}$ is a factor). For a more detailed account of central smoothness and the global dimension lattice, see [3, Sections 5 and 6].

Let Γ be the set of all functions from the positive integers into the discrete set $\{0, 1\}$; under the product topology and pointwise addition modulo 2, Γ is a compact abelian group. Let μ be normalized Haar measure on Γ and let Δ be the dense subgroup of Γ consisting of all those α in Γ with $\alpha(n) = 0$ except for finitely many n. Recall that Woods has constructed a Borel map $\alpha \mapsto \mathfrak{A}(\alpha)$ from Γ into fac$(\mathscr{H})$ such that $\mathfrak{A}(\alpha)$ and $\mathfrak{A}(\beta)$ are spatially isomorphic if and only if $\alpha + \beta \in \Delta$ [6]. Let $\mathfrak{A}$ be the direct integral of the field $\alpha \mapsto \mathfrak{A}(\alpha)$ over Γ with respect to μ. For any Borel subset W of Γ, let $E(W)$ denote the corresponding central projection of $\mathfrak{A}$.

THEOREM. *$\mathfrak{A}$ is a global factor of global type* II_1.

Proof. Let E be a central projection of $\mathfrak{A}$ with $0 < E < I$, and let W be a Borel subset of Γ with $E = E(W)$. Then $0 < \mu(W) < 1$. It is a consequence of part of the proof of [5, Theorem 7.2] that there is an α in Δ such that $W + \alpha$ meets $\Gamma - W$ in a set of positive μ-measure. Then

$$U = W \cap ((\Gamma - W) + \alpha)$$

and

$$V = (W + \alpha) \cap (\Gamma - W)$$

are Borel sets of the same positive μ-measure, and the restriction to U of the map $\beta \mapsto \beta + \alpha$ is a Borel isomorphism of U onto V. One can now build a spatial isomorphism of $\mathfrak{A}_{E(U)}$ onto $\mathfrak{A}_{E(V)}$ by the Mackey–von Neumann cross section theorem [5, Theorem 6.3]. Thus $E(U)$ and $E(V)$ are globally equivalent nonzero subprojections of E and $I - E$, resp. This shows that no nontrivial central projection of $\mathfrak{A}$ can be in the global center of $\mathfrak{A}$, and hence that $\mathfrak{A}$ must be a global factor.

$\mathfrak{A}$ cannot be of global type I since its center contains no minimal projections. So to show that $\mathfrak{A}$ is of global type II_1 it is sufficient to show that it is globally finite. Suppose that E is a central projection of $\mathfrak{A}$ which is globally equivalent to the identity. By [1, p. 212, Théorème 4, and p. 173, Proposition 1(ii)], there must be Borel subsets W and U of Γ with $E = E(W)$ and $\mu(U) = 1$, and a Borel isomorphism f of W onto U such that $f(\beta) + \beta \in \Delta$ for all $\beta \in W$. The sets

$$W(\alpha) = \{\beta \in W : f(\beta) + \beta = \alpha\}, \qquad \alpha \in \Delta,$$

are mutually disjoint Borel subsets of W whose union in W. Notice that

$$\begin{aligned} \mu(W) &= \sum_{\alpha \in \Delta} \mu(W(\alpha)) \\ &= \sum_{\alpha \in \Delta} \mu(W(\alpha) + \alpha) \\ &= \sum_{\alpha \in \Delta} \mu(f(W(\alpha))) \\ &= \mu(U). \end{aligned}$$

Thus $E = I$, showing that $\mathfrak{A}$ must be globally finite. This completes the proof of the theorem.

It is not hard to show that for any two Borel subsets U and V of Γ, the central projections $E(U)$ and $E(V)$ of $\mathfrak{A}$ are globally equivalent if and only if $\mu(U) = \mu(V)$. So the value of the normalized global trace of the global dimension lattice of $\mathfrak{A}$ [4, Theorem 1] at the central projection E of $\mathfrak{A}$ is just $\mu(W)$, where W is any Borel set in Γ with $E = E(W)$.

COROLLARY. *$\mathfrak{A}$ is not centrally smooth.*

Proof. One way of proving the corollary is simply to appeal to the theorem and [3; Section 6]. A second, more direct, proof (the details of which are omitted) may be based on [5, Theorem 7.2].

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